Advanced ODE-Lecture 9
Lyapunov Stability of Autonomous Systems

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Motivation

- Lyapunov method is one of the most important tools in nonlinear systems. It is extremely important in analysis of stability of autonomous systems. It works not only for local, but also for global.

- Lyapunov method has a widespread use in mathematics, control sciences, engineering, physics, financial economics, etc.

Lyapunov Stability for Autonomous Systems

Consider an autonomous system

$$\dot{x} = f(x), \tag{9.1}$$

where $f: D \to R^n$ is locally Lip., $D \subseteq R^n$ and $f(0) = 0$.

We say that the system (9.1) is **complete** if for any initial state $x_0 \in D$, the solution $x = x(t; x_0)$ of (9.1) exists for all $t \geq 0$. That is, there is no **blow-up** for any $x_0 \in D \subseteq R^n$.

1) Statement of Lyapunov Theorem for AS

Theorem 9.1 Let $V : D \to R$ be of C^1 such that

$$V(0) = 0 \quad \text{and} \quad V(x) > 0 \quad \text{in} \quad D - \{0\}; \tag{9.2a}$$

$$V'(x) \overset{def.}{=} \frac{\partial V}{\partial x} \cdot f(x) \leq 0 \quad \text{in} \quad D. \tag{9.2b}$$

Then, $x = 0$ is stable. Moreover, if

$$V'(x) < 0 \quad \text{in} \quad D - \{0\}, \tag{9.2c}$$

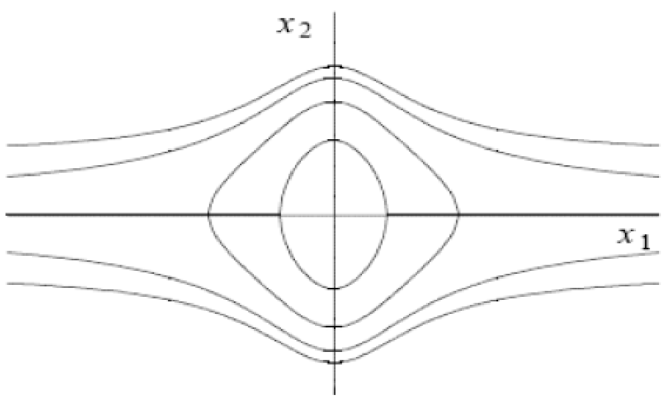
then $x = 0$ is asymptotically stable.

Remark 9.1 $V(x)$ that satisfies (9.2a) is said **positive definite**.

2) Interpretation of Lyapunov conditions

Fact 1. If $V(x) > 0$, then there exists $c^* > 0$ by continuous property such that for all $c \in (0, c^*)$, then the set $V_c := \{x \in R^n \mid V(x) = c\}$ is a compact set encircling the origin, which is said to be **a Lyapunov surface**;

Remark 9.2 If $V(x) > 0$, we can't conclude that for any $c > 0$, V_c is compact. For example, $V(x_1, x_2) = x_2^2 + \dfrac{x_1^2}{1 + x_1^2} > 0$; for $0 < c < 1$, $V(x_1, x_2) = c$ is compact; and for $c > 1$, $V(x_1, x_2) = c$ is not compact. See Fig. 9.1



Fact 2. The derivative of $V(x)$ along trajectories (or solutions) of the system (9.1) is:

$$V'(x) = \frac{\partial V}{\partial x} \cdot f(x),$$

which is an inner product of the gradient of V, **a normal direction** at x of a Lyapunov surface, and $f(x)$, **a tangent direction** at the same point x of the same Lyapunov surface; i.e. **cosine of the included angle** of such two particular vectors.

Fact 3. If $V'(x) < 0$, i.e. cosine of the included angle of above two particular vectors is within $(\frac{\pi}{2}, \pi)$, then, the trajectories move inside the Lyapunov surface V_c.

3) Proof of Lyapunov Theorem for AC

Stability:

Step 1. Given $\varepsilon > 0$, choose $r \in (0, \varepsilon]$ such that

$$B_r = \{x \in R^n \mid \|x\| \le r\} \subset D.$$

Step 2. Let $\alpha = \min_{\|x\|=r} V(x)$. Then, $\alpha > 0$ since $V(x) > 0$, $x \ne 0$. Take $\beta \in (0, \alpha)$, and let

$$\Omega_\beta = \{x \in B_r \mid V(x) \le \beta\}.$$

Then, Ω_β is in the interior of B_r by definition.

Step 3. Ω_β is invariant because $V(x(t)) \le V(x_0) \le \beta$ for all $t \ge 0$ by (9.2b).

Step 4. Ω_β is compact because it is closed by definition and bounded since $\Omega_\beta \subset B_r$. Hence, the system (9.1) has a unique solution for all $t \ge 0$ whenever $x_0 \in \Omega_\beta$ by Extensibility Theorem.

Step 5. Since $V(x)$ is continuous and $V(0)=0$, there is $\delta>0$ such that

$$\|x\| \le \delta \;\Rightarrow\; V(x) < \beta .$$

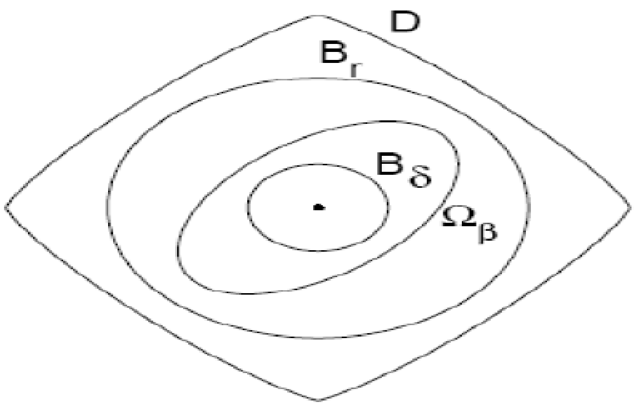
Then, $B_\delta \subset \Omega_\beta \subset B_r$ and

$$x_0 \in B_\delta \;\Rightarrow\; x_0 \in \Omega_\beta \;\Rightarrow\; x(t) \in \Omega_\beta \;\Rightarrow\; x(t) \in B_r .$$

Therefore, for any given $\varepsilon > 0$, there exists $\delta > 0$ s.t.

$$\|x_0\| < \delta \;\Rightarrow\; \|x(t)\| < r \le \varepsilon \text{, for all } t \ge 0 .$$

$x = 0$ is stable by definition. See Fig. 9.2.



Attractivity:

To show $\lim_{t \to +\infty} x(t) = 0$, we need to show that for any $\varepsilon > 0$, there exists $T > 0$ s.t. $\| x(t) \| \le \varepsilon$ for all $t \ge T$.

In the proof of stability, we have shown that for any $\varepsilon > 0$ with B_ε in D, there exists $\eta > 0$ such that $\Omega_\eta \subset B_\varepsilon$, while Ω_η is invariant.

For any $x_0 \in \Omega_\eta$, $x(t) \in \Omega_\eta$ for all $t \ge 0$. Since $V(x(t))$ is monotonically decreasing by (9.2c) and bounded below by zero. Therefore, $\lim_{t \to +\infty} V(x(t)) = c \ge 0$ exists.

Now we show $c = 0$. Otherwise, we suppose $c > 0$. Since $V(x)$ is continuous, there exists $d > 0$ s.t. $B_d \subset \Omega_c \subseteq \Omega_\eta$. Then, since $\lim_{t \to +\infty} V(x(t)) = c > 0$, there exists $T > 0$ such that $x(t)$ lays outside the ball B_d for all $t \ge T$.

Let $-\gamma = \max_{d \le \|x\| \le \eta} V'(x)$, which exists because $V'(x)$ is continuous and has a maximum over the compact set $\{d \le \| x \| \le \eta\}$. By (9.2c), $-\gamma < 0$, and this implies

$$V(x(t)) = V(x_0) + \int_0^t \dot{V}(x(s))\,ds \le V(x_0) - \gamma t, \quad t \ge T.$$

For $t >> T$, $V(x(t)) < 0$. This contradicts $V(x) \ge 0$ for all x. Therefore, we have $\lim_{t \to +\infty} V(x(t)) = 0$, which implies $\lim_{t \to +\infty} x(t) = 0$ by (9.2a). □

Remark 9.3 Theorem 9.1 is a local result. For a global case, we need an additional condition to make sure that Lyapunov surface $V_c := \{x \in R^n \mid V(x) = c\}$ is compact (closed and bounded in R^n) for any $c > 0$ without the breaking.

4) Lyapunov Theorem for GAS

Theorem 9.2 (Barbashin-Krasovskii Theorem) Let $V: R^n \to R$ be of C^1 such that

$$V(0) = 0 \text{ and } V(x) > 0, \ \forall x \neq 0; \tag{9.3a}$$

$$\|x\| \to \infty \ \Rightarrow \ V(x) \to \infty; \tag{9.3b}$$

$$V'(x) < 0, \ \forall x \neq 0. \tag{9.3c}$$

Then, $x = 0$ is globally asymptotically stable (GAS in short).

Proof. For any $x \in R^n$, denote $c = V(x) > 0$. Since $\lim_{\|x\| \to \infty} V(x) = \infty$, by definition of this limit, it implies that for such a $c > 0$, there exists $r > 0$ s.t. $|V(x)| > c$ $\Leftrightarrow V(x) > c$, whenever $\|x\| > r$. This means that $\Omega_c \subset B_r$, which implies that Ω_c is bounded. Then, the rest of the proof is similar to that of Theorem 9.1. □

Remark 9.4 The condition (9.3b) is said to be **radially unbounded**. If the system (9.1) is GAS, then, the equilibrium must be unique. **(why!)**

5) Lyapunov Theorem for Unstability

Two facts:

Fact 1. $V(x) > 0$ plus $\dot{V}(x) > 0 \Rightarrow$ the origin is unstable.

Fact 2. When testing instability, the above conditions can be relaxed.

Theorem 9.3 (Chetaev Theorem) Let $V: D \rightarrow R$ be of C^1 s.t. $V(0) = 0$ and $V(x_0) > 0$ for some x_0 with arbitrarily small $\| x_0 \|$. Define $U = \{x \in B_r \mid V(x) > 0\}$ and suppose that $V'(x) > 0$ in U. Then, $x = 0$ is unstable.

Proof. Since $V(x_0) = a > 0$, so $x_0 \in U$. $x(t)$ starting at $x(0) = x_0$ will leave U.

To see this point, if $x(t) \in U \;\; \Rightarrow \;\; V(x(t)) \geq a$, since $V'(x) > 0$ in U. Let

$$\gamma = \min\{V'(x) \mid x \in U \text{ and } V(x) \geq a\},$$

which exists since the continuous function $V'(x)$ has a minimum over the compact set $\{x \in U$ and $V(x) \geq a\}$. Then, $\gamma > 0$ and

$$V(x(t)) = V(x_0) + \int_0^t V'(x(s))ds \geq a + \int_0^t \gamma ds = a + \gamma t.$$

This inequality shows that $\lim_{t \to +\infty} V(x(t)) = \infty$. It implies that $x(t)$ cannot stay forever in U because $V(x)$ is bounded on U. Now, $x(t)$ cannot leave U through the surface $V(x) = 0$ since $V(x(t)) \geq a$ for all $t \geq 0$. Hence, it must leave U through the sphere $\| x \| = r$. Since it can happen for an arbitrarily small $\| x_0 \|$, the origin is unstable. □

Remark 9.5 Since U is not necessarily a neighborhood of the origin, then

1) $V(x)$ in Chetaev Theorem does not have to be positive definite!

2) $V'(x)$ in Chetaev Theorem does not have to be positive definite!

Some Examples

Example 9.1 Consider

$$\begin{cases} x_1' = ax_1 - x_2 + kx_1(x_1^2 + x_2^2) \\ x_2' = x_1 - ax_2 + kx_1(x_1^2 + x_2^2) \end{cases}, \tag{9.4}$$

where $a > 0$, $a \neq 1$, and k is a parameter. Clearly, the origin is equilibrium. The linearization gives

$$A = \begin{pmatrix} a & -1 \\ 1 & -a \end{pmatrix}$$

with $\lambda = \pm\sqrt{a^2 - 1}$ and $g_1(x_1, x_2) = g_2(x_1, x_2) = kx_1(x_1^2 + x_2^2)$ satisfying

$$\lim_{\|x\| \to \infty} \frac{\| g(x) \|}{\| x \|} = 0. \tag{9.5}$$

If $a > 1$, the origin is a saddle point, which is unstable. Then, (9.4) is also unstable by linearization.

If $0 < a < 1$, the origin is a center, which is stable but not AS. The linearization fails this time. However, the linearized system has the equation for trajectories given by

$$\frac{dx_2}{dx_1} = \frac{x_1 - ax_2}{ax_1 - x_2},$$

whose general solution is solved by

$$x_1^2 - 2ax_1x_2 + x_2^2 = \tilde{c}.$$

These trajectories are ellipses if $\tilde{c} > 0$. The trajectory is the origin if $\tilde{c} = 0$ (See Remark 9.6) So it can be taken as a Lyapunov function candidate for the nonlinear system (9.4)

$$V(x_1, x_2) = x_1^2 - 2ax_1x_2 + x_2^2,$$

which is positive definite. Taking derivative along trajectories of (9.4) results in

$$V'(x_1, x_2) = 2k(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 2ax_1x_2).$$

Then, $\dot{V}(x_1, x_2) < 0$ if $k < 0$ and $V'(x_1, x_2) > 0$ if $k > 0$. We conclude that (9.4) is AS if $k < 0$ by Theorem 9.1 and it is unstable if $k > 0$ by Theorem 9.3. Moreover, (9.4) is GAS because $V_c(x_1, x_2) = \{(x_1, x_2) \mid x_1^2 - 2ax_1x_2 + x_2^2 = \tilde{c} > 0\}$ is a Lyapunov surface of ellipses for all $\tilde{c} > 0$, which clearly satisfy the radially unbounded condition (9.3b).

Remark 9.6 The general conic equation is given by

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0, \tag{9.6}$$

where A, B and C are not all zero. If $\Delta_1 \cdot \Delta_3 < 0$ and $\Delta_2 > 0$, then (11.6) is an ellipse, where

$$\Delta_1 = A + C, \quad \Delta_2 = \begin{vmatrix} A & B \\ C & D \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix}.$$

In Example 9.1, $A = 1$, $B = -a$, $C = 1$, $D = 0$, $E = 0$ and $F = -\tilde{c}$. It is easy to be verified that when $0 < a < 1$, $x_1^2 - 2ax_1x_2 + x_2^2 = \tilde{c}$ for all $\tilde{c} > 0$ are ellipses.

Example 9.2 Consider the pendulum equation without friction:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{g}{l}\sin x_1 \end{cases}.$$

Take the energy function

$$V(x) = \frac{g}{l}(1-\cos x_1) + \frac{1}{2}x_2^2 .$$

Clearly, $V(0)=0$ and $V(x)>0$ is over $-2\pi < x_1 < 2\pi$.

$$V'(x) = \frac{g}{l}\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = \frac{g}{l}x_2 \sin x_1 - \frac{g}{l}x_2 \sin x_1 = 0 .$$

$\Rightarrow$ The origin is stable. Since $V'(x) \equiv 0$, $\Rightarrow$ $V(x(t)) \equiv c > 0$ $\Rightarrow$ $\lim_{t\to\infty} x(t) \neq 0$, the origin is not AS.

Example 9.3 Consider the pendulum equation with friction:

$$\begin{cases} x_1' = x_2 \\ x_2' = -\dfrac{g}{l}\sin x_1 - \dfrac{k}{m}x_2 \end{cases} \tag{9.7}$$

Let us try again $V(x) = \dfrac{g}{l}(1-\cos x_1) + \dfrac{1}{2}x_2^2$. Since

$$V'(x) = \frac{g}{l}x_1' \sin x_1 + x_2 x_2' = -\frac{k}{m}x_2^2 \le 0,$$

$\Rightarrow$ The origin is stable only. However, the experience tells that it is AS because of the friction.

Remark 9.9 We may apply the finer Lyapunov function to (9.7) as follows.

$$V(x) = \frac{1}{2}x^T Px + \frac{g}{l}(1-\cos x_1) = \frac{1}{2}(x_1, x_2)\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{g}{l}(1-\cos x_1),$$

where P is positive definite. Try to determine the elements p_{ij} of P such that

$V'(x) < 0$. (**Homework**).

Example 9.4 Consider the system

$$\begin{cases} \dot{x}_1 = x_1 + g_1(x) \\ \dot{x}_2 = -x_2 + g_2(x) \end{cases},$$

where $g_1(x)$ and $g_2(x)$ satisfy $|g_j(x)| \le k \|x\|^2$ near the origin. Consider the function $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$. On the line $x_2 = 0$, $V(x) > 0$ at points arbitrarily close to the origin. The derivative of $V(x)$ is given by

$$V'(x) = x_1^2 + x_2^2 + x_1 g_1(x) - x_2 g_2(x).$$

Since $|x_1 g_1(x) - x_2 g_2(x)| \le \sum_{j=1}^{2} |x_j| \cdot |g_j(x)| \le 2k \|x\|^3$. Hence,

$$V'(x) \ge \|x\|^2 - 2k\|x\|^3 = \|x\|^2 (1 - 2kr\|x\|).$$

Choosing r such that $B_r \subset D$ and $r < \frac{1}{2k}$, the origin is unstable by Chetaev Theorem.

Summary

- Theorem 9.1-9.3 consist of the classical Lyapunov theory. LaSalle-Krosovskii Theorem is the starting of the modern Lyapunov theory.

- GAS is more interesting for engineering application because it is no need for the estimation of a region of attraction, which is usually a tough work. However, GAS requirement is more demanding. In control, people hope to get (robustly) globally asymptotical stabilization by feedback (refer to feedback control), or moreover, to meet some additional optimized condition (refer to optimized control).

Homework

1. Study the stability of the pendulum equation with friction

$$\begin{cases} x_1' = x_2 \\ x_2' = -\dfrac{g}{l}\sin x_1 - \dfrac{k}{m}x_2 \end{cases}$$

by linearization.

2. Do Remark 9.7.